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Alternating Parity of Tchebycheff Systems

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A system of functions $\{U_k\}$, $U_k \in C[a, b]$, $k = 0, 1,..., n$, is said to be a Tchebycheff system (TS) on $[a, b]$ if $\sum_{k=0}^{n} a_k U_k$ has at most n distinct zeros in $[a, b]$, for any nontrivial choice of real $\{a_k\}$. Let $0 = t_0 < t_1 < \cdots < t_n$ be a sequence of real numbers. It is known that $\{x^{t_k}\}$, $k = 0, 1,..., n$, is a TS on any interval $[a, b]$ satisfying $0 \leqslant a < b$ [1]. If $a < 0 < b$, this result does not necessarily hold; e.g., $\{1, x^2\}$ is not a TS on $[-1, 1]$. In this paper we seek conditions on $\{t_k\}$ that are necessary and sufficient for $\{x^{t_k}\}$ to be a TS on $(-\infty, \infty)$. In this case, naturally, we restrict the t_k to be integers.

DEFINITION. A sequence of integers $\{t_k\}_{k=0}^{n}$ is said to have the alternating parity property (APP) if and only if $t_0 = 0$ and for all k, t_{2k} is even and t_{2k+1} is odd.

THEOREM. *Let $0 = t_0 < t_1 < \cdots < t_n$ be a sequence of integers. Then $\{x^{t_k}\}$ is a TS on $(-\infty, \infty)$ if and only if $\{t_k\}$ has APP.*

Proof. Suppose $\{t_k\}$ has APP. We prove by induction that $\{x^{t_k}\}$ is a TS on $(-\infty, \infty)$.

$\{0, t_1\}$ has APP if and only if t_1 is odd, in which case $a_0 + a_1 x^{t_1}$ has a unique zero for any nontrivial choice of a_0, a_1. Assume the theorem true for $n - 1$. Let $\{a_k\}$, $k = 0,1,..., n$ be arbitrary and set $P(x) = \sum_{k=0}^{n} a_k x^{t_k}$. Then $P'(x) = \sum_{k=1}^{n} (a_k t_k)\, x^{t_k - 1} = x^{t_1 - 1} \sum_{k=1}^{n} (a_k t_k)\, x^{t_k - t_1}$.

Let $b_{k-1} = a_k t_k$ and $S_{k-1} = t_k - t_1$, $k = 1, 2,..., n$. Then $S_0 = 0$ and $\{S_k\}$, $k = 0, 1,..., n - 1$ has APP. Thus, $P'(x) = x^{t_1 - 1} \sum_{k=0}^{n-1} b_k x^{S_k} = x^{t_1 - 1} Q(x)$. By induction hypothesis $Q(x)$ has at most $n - 1$ distinct zeros. If $b_0 = 0$, then $P'(x)$ has at most $n - 1$ distinct zeros, so that $P(x)$ has at most n distinct zeros, since the zeros of $P'(x)$ either separate the roots of $P(x)$, are points of inflection of $P(x)$, or are zeros of $P(x)$. If $b_0 \neq 0$, then 0 is not a root of $Q(x)$, so that $P'(x)$ could have n zeros. Since, however, $t_1 - 1$ is even, 0 is a point of inflection of $P(x)$, and not a local maximum or minimum.

0 is thus not a separating zero of $P'(x)$, so that $P'(x)$ has at most $n-1$ distinct separating zeros, and, hence, $P(x)$ has at most n distinct zeros.

Before considering the converse we prove a preliminary result.

LEMMA. *Suppose* $\{t_k\}$, $k = 0, 1, \ldots, n$, *has APP. Then there exist* $\{a_k\}$, $k = 0, 1, \ldots, n$ *such that* $P(x) = \sum_{k=0}^{n} a_k x^{t_k}$ *has n simple real zeros.*

Proof. By induction. $\{0, t_1\}$ has APP if and only if t_1 is odd. In this case, $P(x) = 1 - x^{t_1}$ has a simple zero at $x = 1$.

Assume the lemma is true for $n-1$; i.e., suppose there exist $\{a_k\}$, $k = 0, 1, \ldots, n-1$ such that $Q(x) = \sum_{k=0}^{n-1} a_k x^{t_k}$ has $n-1$ simple real zeros. We may assume that $a_{n-1} > 0$.

Case 1. t_{n-1} odd, t_n even. Then $Q(x) \to -\infty$ as $x \to -\infty$. Let

$$m = \min\{|\, Q(x)|: x \text{ a local max or min of } Q(x)\};$$

$$a = \min\{x: Q(x) = -m\};$$

$$b = \max\{x: Q(x) = m\};$$

$$c = \max(|\, a\,|, |\, b\,|);$$

$$\epsilon = m/2c^{t_n}.$$

Note that $m > 0$, since, by induction hypothesis, $Q(x)$ is assumed to have $n-1$ simple real zeros, while by the first part of the theorem, $Q(x)$ can have no more than $n-1$ distinct real zeros. Thus $Q(x)$ has exactly $n-1$ simple real zeros and no other real zeros, so that if $Q(\xi) = 0$ then $Q'(\xi) \neq 0$. Hence, if $Q(\xi) = 0$, then ξ is not a local max or min of $Q(x)$.

Let $P(x) = Q(x) + \epsilon x^{t_n}$. Then $|\, P(x) - Q(x)| \leqslant m/2$, $a \leqslant x \leqslant b$. Let x_i, $i = 1, 2, \ldots, n-1$, be the zeros of $Q(x)$. Then there exist $z_i \in (x_i, x_{i+1})$, $i = 1, 2, \ldots, n-2$, such that $Q'(z_i) = 0$. Then $Q(a) = -m$, $Q(z_1) \geqslant m$, $Q(z_2) \leqslant -m, \ldots, Q(z_{n-2}) \leqslant -m$. Thus $P(a) < 0$, $P(z_1) > 0$, $P(z_2) < 0, \ldots$, $P(z_{n-2}) < 0$, so that $P(x)$ has $n-2$ zeros in (a, z_{n-2}). Also $P(x) \to \infty$ as $x \to \pm\infty$, and, hence, $P(x)$ has two additional zeros, one in $(-\infty, a)$, the other in (z_{n-2}, ∞). Thus $P(x)$ has a total of n simple real zeros.

Case 2. t_{n-1} even, t_n odd. Then $Q(x) \to \infty$ as $x \to -\infty$.
Let m, b be as in Case 1, and let

$$a = \min\{x: Q(x) = m\};$$

$$c = \max(|\, a\,|, |\, b\,|);$$

$$\epsilon = m/2c^{t_n},$$

and let $P(x) = Q(x) + \epsilon x^{t_n}$. Following the method of Case 1, we prove that $P(x)$ has n simple real zeros, thus proving the lemma.

We return to the converse of the theorem which is also proved by induction. $\{0, t_1\}$ does not have APP if and only if t_1 is even. In this case $P(x) = 1 - x^{t_1}$ has two distinct real zeros in $(-\infty, \infty)$ so that $\{1, x^{t_1}\}$ is not a TS on $(-\infty, \infty)$.

Now suppose the theorem is true for $n - 1$, and suppose $\{t_k\}$, $k = 0, 1,..., n$ does not have APP.

Case 1. $\{t_k\}$, $k = 0, 1,..., n - 1$ has APP.

There are two possibilities: t_{n-1} and t_n odd, or t_{n-1} and t_n even. We consider only the first; the second is handled analogously. By the lemma, there exist $\{a_k\}$, $k = 0, 1,..., n - 1$ such that $Q(x) = \sum_{k=0}^{n-1} a_k x^{t_k}$ has $n - 1$ simple real zeros. We assume that $a_{n-1} > 0$. Then $Q(x) \to \infty$ as $x \to \infty$ and $Q(x) \to -\infty$ as $x \to -\infty$. Let m, a, b, c, and ϵ be as in Case 1 of the lemma, and let $P(x) = Q(x) - \epsilon x^{t_n}$. Let x_i, $i = 1, 2,..., n - 1$ be the zeros of $Q(x)$. As in the lemma, there exist $z_i \in (x_i, x_{i+1})$, $i = 1, 2,..., n - 2$, such that $Q'(z_i) = 0$. Again, $P(a) < 0$, $P(z_1) > 0$, $P(z_2) < 0,..., P(z_{n-2}) < 0$, $P(b) > 0$. Thus, $P(x)$ has $n - 1$ roots in (a, b). But $P(x) \to -\infty$ as $x \to \infty$ and $P(x) \to \infty$ as $x \to -\infty$, and, hence, $P(x)$ has at least two additional zeros, one in $(-\infty, a)$ and another in (b, ∞), for a total of at least $n + 1$ distinct zeros. Thus $\{x^{t_k}\}$, $k = 0, 1,..., n$, is not a TS on $(-\infty, \infty)$.

Case 2. $\{t_k\}$, $k = 0, 1,..., n - 1$ does not have APP.

Then, by induction hypothesis, there exist $\{a_k\}$, $k = 0, 1,..., n - 1$, such that $Q(x) = \sum_{k=0}^{n-1} a_k x^{t_k}$ has n simple real zeros. Once again it is necessary to distinguish between various possibilities of the parities of t_{n-1} and t_n, as in Case 1. In each case, however, the appropriate choice of either $P(x) = Q(x) + \epsilon x^{t_n}$ or $P(x) = Q(x) - \epsilon x^{t_n}$ will guarantee that $P(x)$ has at least $n + 1$ distinct real zeros. The proof is now complete.

A system of functions $\{U_k\}$, $U_k \in C[a, b]$, $k = 0, 1,..., n$ is said to be an interpolation system on $[a, b]$ if for any real $\{x_k\}$, $\{y_k\}$, $a \leqslant x_k \leqslant b$, $k = 0, 1,..., n$, there exist $\{a_k\}$, $k = 0, 1,..., n$ such that $\sum_{k=0}^{n} a_k U_k(x_j) = y_j$, $j = 0, 1,..., n$.

It is well known that $\{U_k\}$ is an interpolation system on $[a, b]$ if and only if $\{U_k\}$ is a TS on $[a, b]$ [2]. We can thus conclude with an interpolation theorem stated as a

COROLLARY. *Let* $0 = t_0 < t_1 < \cdots < t_n$ *be a sequence of integers. Then* $\{x^{t_k}\}$ *is an interpolation system on* $(-\infty, \infty)$ *if and only if* $\{t_k\}$ *has APP.*

References

1. E. W. Cheney, "Introduction to Approximation Theory," p. 77, McGraw–Hill, New York, 1966.
2. G. G. Lorentz, "Approximation of Functions," p. 24, Holt, Rinehart, and Winston, New York, 1966.

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